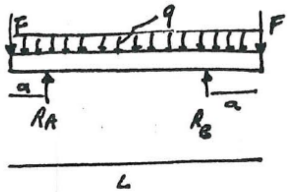
Flexion, Exercice 7



• Réactions en A et B

$$R_A = R_B = \frac{qL}{2} + F$$

• $\vec{M}$

• $0<x<a$ $\quad M = -qx \cdot \frac{x}{2} - F \cdot x = \frac{-qx^2}{2} - Fx$

• $a<x<L-a$ $\quad M = -qx \cdot \frac{x}{2} + R_A(x-a) - Fx = -\frac{q}{2}(x^2 - Lx + La) - aF$

• M_{min} en A et B, $x=a$, $M_{min} = \frac{-qa^2}{2} - aF$

• M_{max} à $x=\frac{L}{2}$, $M_{max} = \frac{qL}{2}\left(\frac{L}{4} - a\right) - aF$

• $|M_{min}| = |M_{max}|$

$$\Rightarrow \frac{qa^2}{2} + aF = \frac{qL}{2}\left(\frac{L}{4} - a\right) - aF$$

$$a^2\left(\frac{q}{2}\right) + a\left(2F + \frac{qL}{2}\right) - \frac{qL^2}{8} = 0$$

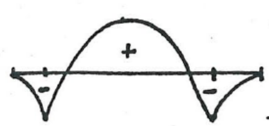
ax^2+bx+c, $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$a = \frac{1}{q}\left[-\left(2F + \frac{qL}{2}\right) \pm \sqrt{\left(2F+\frac{qL}{2}\right)^2 + \frac{4q^2L^2}{16}}\right]$$

seule la solution + à un sens

• $L = 1{,}5\ m$, $F = 30'000$, $q = 2000\ N/cm$ $\Rightarrow a = 0{,}24\ m$

•



$\rightarrow M_{max} = 13'000\ Nm$

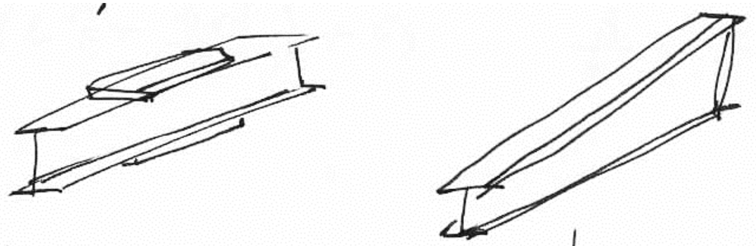
$\rightarrow M_{min} = 13'000\ Nm$

• $h = ?$ avec $b = 4\ cm$ et $\sigma_{max} = 150\ MPa$

$$\sigma_{max} = \frac{M_{max}}{bh^3/12} \cdot \frac{h}{2} = \frac{6 M_{max}}{bh^2}$$

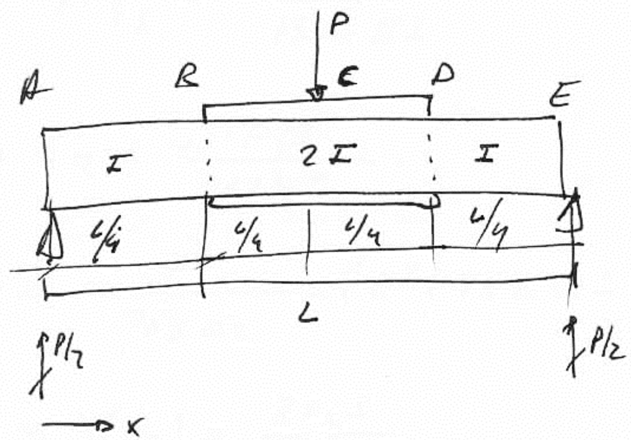
$$h = \sqrt{\frac{6 \cdot M_{max}}{b \cdot \sigma_{max}}} = \sqrt{\frac{6 \cdot 13'000}{4 \cdot 10^{-2} \times 150 \cdot 10^6}} = 11{,}4 \cdot 10^{-2}\ m$$

Poutres non-prismatiques



→ I ↑ localement où M est élevé

Méthode numérique par les itérations



$M = \frac{P.x}{2} \qquad 0 < x < \frac{L}{2}$

$0 < x < \frac{L}{4} \qquad EIy'' = \frac{Px}{2}, \quad y' = \frac{Px^2}{4EI} + C_1$

$\frac{L}{4} < x < \frac{L}{2} \qquad E2Iy'' = \frac{Px}{2}, \quad y' = \frac{Px^2}{8EI} + C_2$

P.-Etienne Bourban

Poutres non-prismatiques

conditions aux bords ⇒ csts d'intégration

1. en A : $x = 0 \Rightarrow y = 0$
2. en C : $x = \frac{L}{2}$, $y' = 0$
3. en B : $x = \frac{L}{4}$, pente de AB = pente de BC
4. en B : $x = \frac{L}{4}$, $y_{AB} = y_{BC}$

conditions continuité

$$② \Rightarrow C_2 = -\frac{PL^2}{32EI}$$

$$\Rightarrow y' = -\frac{P}{32EI}(L^2 - 4x^2) \quad \Rightarrow y'\left(\frac{L}{4}\right) = \text{pente en B} = -\frac{3PL^2}{128EI}$$

$$③ \Rightarrow \frac{P}{4EI}\left(\frac{L}{4}\right)^2 + C_1 = -\frac{3PL^2}{128EI} \quad \Rightarrow C_1 = -\frac{5PL^2}{128EI}$$

$$\Rightarrow y'_{AB} = -\frac{P}{128}(5L^2 - 32x)$$

en $x = 0$ → $\theta_A = -y'(0) = \frac{5PL^2}{128EI}$

Poutres non-prismatiques

$$y = -\frac{P}{128\,EI}\left(5L^2x - \frac{32x^3}{3}\right) + C_3 \qquad 0 < x < \frac{L}{4}$$

$$y = -\frac{P}{32\,EI}\left(L^2x - \frac{4x^3}{3}\right) + C_4 \qquad \frac{L}{4} < x < \frac{L}{2}$$

$$(1) \Rightarrow C_3 = 0$$

$$y = -\frac{Px}{384\,EI}\left(15L^2 - 32x^2\right)$$

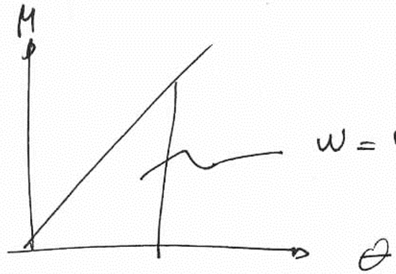
$$y\left(\frac{L}{4}\right) = -\frac{13\,PL^3}{1536\,EI}$$

$$(4) \Rightarrow C_4 = -\frac{PL^3}{768\,EI}$$

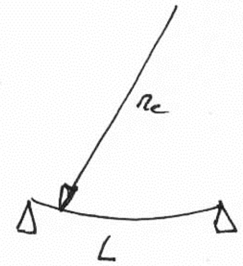
$$\Rightarrow y = -\frac{P}{768\,EI}\left(L^3 + 24L^2x - 32x^3\right) \qquad \frac{L}{4} < x < \frac{L}{2}$$

$$y_c = y\left(\frac{L}{2}\right) = \frac{3PL^3}{256\,EI}$$

Energie de déformation



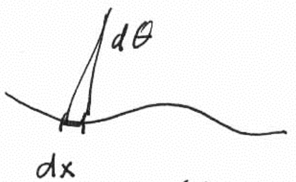
$$W = U = \frac{M\theta}{2}$$



$$\theta = \frac{L}{r_c} = \kappa L = \frac{ML}{EI} \qquad M = \text{cste}$$

$$\Rightarrow U = \frac{M^2 L}{2EI}$$

$$\text{ou} \quad U = \frac{EI\theta^2}{2L}$$



$$d\theta = \kappa\, dx = \frac{d^2y}{dx^2}\, dx$$

$$dU = \frac{M^2 dx}{2EI}$$

$$dU = \frac{EI(d\theta)^2}{2dx} = \frac{EI}{2dx}\left(\frac{d^2y}{dx^2}\, dx\right)^2 = \frac{EI}{2}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$U = \int \frac{M^2 dx}{2EI}, \qquad U = \int \frac{EI}{2}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

Théorème de Castigliano

— Single load P

$$U = \frac{P\delta}{2} \qquad U = \frac{M\theta}{2}$$

$$\downarrow \qquad\qquad \downarrow$$

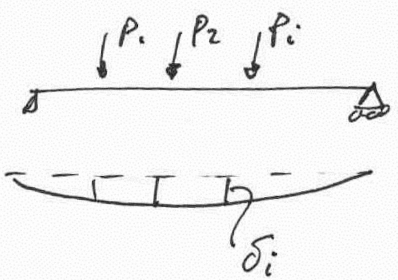
$$\delta = \frac{2U}{P} \qquad \theta = \frac{2U}{M}$$

δ = déflection au point où P est appliqué

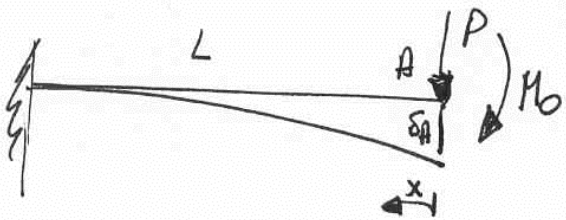
θ = angle rota au point où M est appliqué

— Castigliano

$$\delta_i = \frac{\partial U}{\partial P_i}$$



Castigliano pour les poutres en flexion



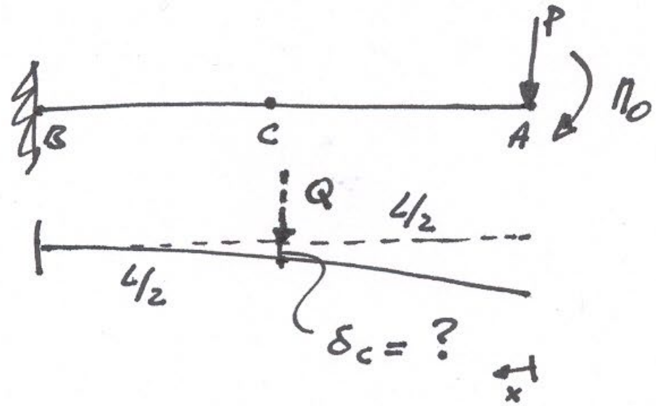
$$M = -Px - M_0$$

$$U = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L (-Px - M_0)^2 dx = \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{M_0^2 L}{2EI}$$

$$\delta_A = \frac{\partial U}{\partial P} = \frac{PL^3}{3EI} + \frac{M_0L^2}{2EI}$$

$$\theta_A = \frac{\partial U}{\partial M_0} = \frac{PL^2}{2EI} + \frac{M_0L}{EI}$$

Méthode de la charge fictive



$0 \leq x \leq \frac{L}{2}$ $\qquad M = -P.x - M_0$

$\frac{L}{2} \leq x \leq L$ $\qquad M = -P.x - M_0 - Q\left(x - \frac{L}{2}\right)$

$$V_{AC} = \int_0^{L/2} \frac{M^2 dx}{2EI} = \frac{1}{2EI} \int_0^{L/2} (-Px - M_0)^2 dx$$

$$= \frac{P^2 L^3}{48EI} + \frac{P.M_0 L^2}{8EI} + \frac{M_0^2 L}{4EI}$$

Méthode de la charge fictive

$$U_{CB} = \int_{L/2}^{L} \frac{M^2 dx}{2EI} = \frac{1}{2EI} \int_{L/2}^{L} \left(-Px - M_0 - Q\left(x - \frac{L}{2}\right)\right)^2 dx$$

$$= \frac{7P^2L^3}{48EI} + \frac{3PM_0L^2}{8EI} + \frac{5PQL^3}{48EI} + \frac{M_0^2 L}{4EI} + \frac{M_0QL^2}{8EI} + \frac{Q^2L^3}{48EI}$$

$$U = U_{AC} + U_{CB} = \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{5PQL^3}{48EI} + \frac{M_0^2 L}{2EI} + \frac{M_0QL^2}{8EI} + \frac{Q^2L^3}{48EI}$$

$$\delta_C = \frac{\partial U}{\partial Q} = \frac{5PL^3}{48EI} + \frac{M_0L^2}{8EI} + \frac{QL^3}{24EI}$$

$$Q = 0$$

$$\delta_C = \frac{5PL^3}{48EI} + \frac{M_0L^2}{8EI}$$

Castigliano modifié

$$\delta_i = \frac{\partial U}{\partial P_i} = \frac{\partial}{\partial P_i} \int \frac{M^2 dx}{2EI} = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P_i}\right) dx$$

Dériver avant d'intégrer

A vous de l'appliquer pour calculer rapidement le déplacement de C de la poutre précédente

Castigliano modifié

$$\delta_i = \frac{\partial U}{\partial P_i} = \frac{\partial}{\partial P_i} \int \frac{M^2 dx}{2EI} = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P_i}\right) dx$$

Dériver avant d'intégrer

$$M = -P.x - M_0 \qquad \frac{\partial M}{\partial Q} = 0 \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$M = -P.x - M_0 - Q\left(x - \frac{L}{2}\right) \qquad \frac{\partial M}{\partial Q} = -\left(x - \frac{L}{2}\right) \qquad \left(\frac{L}{2} \le x \le L\right)$$

$$\delta_C = \frac{1}{EI}\int_0^{L/2} (-Px - M_0) \cdot 0 \, dx + \frac{1}{EI}\int_{L/2}^{L} \left(-Px - M_0 - Q\left(x - \frac{L}{2}\right)\right) \cdot \left(-\left(x - \frac{L}{2}\right)\right) dx$$

$$Q = 0$$

$$\delta_C = \frac{1}{EI}\int_{L/2}^{L} (-Px - M_0) \cdot \left(-\left(x - \frac{L}{2}\right)\right) dx = \frac{5PL^3}{48EI} + \frac{M_0 L^2}{8EI}$$